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Behaviour of the critical wavevector near a Lifshitz point

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Abstract. Using scaling arguments, it is shown that, at a Lifshitz point, the exponent β_k is related to the crossover exponent ϕ by $\beta_k = \nu_{l4}/\phi$. The exponents η_{l4} and β_k are calculated for a uniaxial Lifshitz point to $O(\epsilon_l^2)$, where $\epsilon_l = 4{\cdot}5 - d$. The results are $\eta_{l4} = -[(n+2)/4(n+8)^2]\epsilon_l^2$ and $\beta_k = \frac{1}{2} + [7(n+2)/16(n+8)^2]\epsilon_l^2$.

1. Introduction

A Lifshitz point (Hornreich *et al* 1975) is a multicritical point which divides a second-order phase transition (λ) line into two segments, λ_1 and λ_2, such that on only one of them (λ_1) is the critical order parameter characterised by a fixed equilibrium wavevector. Approaching the Lifshitz point on the helicoidal or λ_2 segment of the λ-line, the magnitude of the critical wavevector $k_c(g)$ is expected to be asymptotically related to the non-ordering field g by (Hornreich *et al* 1975)

$$k_c(g) - k_L \sim (g - g_L)^{\beta_k}; \qquad g > g_L. \tag{1}$$

Here, g_L is the value of g at the Lifshitz point and $k_L \equiv k_c(g_L)$ (henceforth taken to be zero) is the magnitude of the g-independent critical wavevector characterising the order parameter for $g \leq g_L$. In this paper, we present scaling arguments relating β_k to the crossover exponent ϕ at the multicritical point, together with Feynman graph calculations of β_k. We shall concentrate on an analysis of the *uniaxial* Lifshitz point (where the wavevector instability occurs in one dimension only) as this is expected to be the experimentally relevant case. Results for the isotropic Lifshitz point will also be reported, although these are likely to be of purely academic interest since it is believed that the phase transition in this case is of first order (Brazovskii 1975).

2. Scaling theory at a Lifshitz point

We denote by μ_t and μ_δ the linear scaling fields (Fisher 1974) associated with the relevant temperature and crossover variables at a Lifshitz fixed point. Let us consider the two-spin correlation function $G(t, \delta, \boldsymbol{k}, \boldsymbol{q})$, where $\delta \equiv g - g_L$ and $t \equiv T - T_c$ ($\delta = 0$), so that $t = \delta = 0$ defines the Lifshitz critical point. The vectors $\boldsymbol{k}$ and $\boldsymbol{q}$ are m- and $(d-m)$-dimensional wavevector components. The wavevector instability occurs in the

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m-dimensional subspace only. The correlation function G can be written as a function of the scaling fields

$$G(t, \delta, \boldsymbol{k}, \boldsymbol{q}) = \hat{G}(\mu_t, \mu_\delta, \boldsymbol{k}, \boldsymbol{q}). \tag{2}$$

Explicit renormalisation-group analysis (Hornreich *et al* 1975) shows that $\hat{G}$ has homogeneous (scaling) properties such that

$$\hat{G}(\mu_t, \mu_\delta, \boldsymbol{k}, \boldsymbol{q}) = a^{4-\eta_{l4}}\hat{G}(a^{\lambda_t}\mu_t, a^{\lambda_\delta}\mu_\delta, a\boldsymbol{k}, b\boldsymbol{q}), \tag{3}$$

where a, b are scaling lengths, and η_{l4}, λ_t, λ_δ are constants. Taking $\boldsymbol{k} = \boldsymbol{q} = 0$ and $a = \mu_t^{-1/\lambda_t}$ gives

$$\hat{G}(\mu_t, \mu_\delta, 0, 0) = \mu_t^{-(4-\eta_{l4})/\lambda_t}\hat{G}(1, \mu_\delta/(\mu_t^{\lambda_\delta/\lambda_t}), 0, 0). \tag{4a}$$

Identifying $\gamma_l = (4-\eta_{l4})/\lambda_t = (4-\eta_{l4})\nu_{l4}$ and $\phi = \lambda_\delta/\lambda_t$, (4$a$) becomes

$$\hat{G}(\mu_t, \mu_\delta, 0, 0) = \mu_t^{-\gamma_l}X(\mu_\delta/\mu_t^{\phi}). \tag{4b}$$

Now, although the $\boldsymbol{k} = \boldsymbol{q} = 0$ correlation function remains finite at the true critical points t_c on the helicoidal segment of the λ-line, we expect that it will at least carry the usual energy-type singularity (see, e.g., Aharony and Fisher 1973). Thus $X(x)$ must be singular, at $x = \dot{x}$ say, where

$$\mu_\delta(t_c)/(\mu_t(t_c))^{\phi} = \dot{x} \tag{5a}$$

or

$$\bar{\mu}_t \equiv \mu_t(t_c) = (\mu_\delta(t_c)/\dot{x})^{1/\phi} \equiv (\bar{\mu}_\delta/\dot{x})^{1/\phi}. \tag{5b}$$

We next consider the $\boldsymbol{k}$-dependent pair-correlation function at the true critical points (again on the helicoidal segment). We have

$$\begin{aligned}\hat{G}(\bar{\mu}_t, \bar{\mu}_\delta, \boldsymbol{k}, 0) &\\ &= a^{4-\eta_{l4}}\hat{G}(a^{\lambda_t}(\bar{\mu}_\delta/\dot{x})^{1/\phi}, a^{\lambda_\delta}\bar{\mu}_\delta, a\boldsymbol{k}, 0)\\ &= a^{4-\eta_{l4}}\hat{G}((a^{\lambda_\delta}\bar{\mu}_\delta/\dot{x})^{1/\phi}, a^{\lambda_\delta}\bar{\mu}_\delta, a\boldsymbol{k}, 0) = k^{-(4-\eta_{l4})}Y(\bar{\mu}_\delta/k^{\lambda_\delta}),\end{aligned} \tag{6}$$

where the last expression is obtained by setting $a = k^{-1}$ and recognising that $\hat{G}$ is independent of $\hat{\boldsymbol{k}} = \boldsymbol{k}/k$. The function $Y(y)$ must have a singularity, at $y = \dot{y}$ say, which determines the critical wavevector. We have

$$\bar{\mu}_\delta/k_c^{\lambda_\delta} = \dot{y}, \tag{7a}$$

so that

$$k_c \sim \bar{\mu}_\delta^{1/\lambda_\delta}. \tag{7b}$$

In general, we expect μ_t and μ_δ to be linear combinations of the physical fields t and δ. Writing

$$\delta = e_\delta\mu_\delta + e_t\mu_t, \tag{8a}$$

we have from (5b) at $t = t_c$

$$\delta = e_\delta\bar{\mu}_\delta + e_t(\bar{\mu}_\delta/\dot{x})^{1/\phi} \tag{8b}$$

$$\simeq e_\delta\bar{\mu}_\delta \qquad \text{for } \phi < 1. \tag{8c}$$

Thus, for $\phi < 1$, we expect that $k_c \sim \delta^{1/\lambda_\delta}$, and we can identify

$$\beta_k = 1/\lambda_\delta. \tag{9}$$

This will, in fact, prove to be the case of interest here. Note, however, that if $\phi > 1$ *and* $e_t \neq 0$, we have $\delta \sim \bar{\mu}_\delta^{1/\phi}$ so that $k_c \sim \delta^{1/\lambda_t} = \delta^{\nu_{l4}}$ and $\beta_k = \nu_{l4}$.

3. The self-energy

For the uniaxial ($m = 1$) Lifshitz point, we shall consider the Landau–Ginzburg–Wilson effective Hamiltonian (Hornreich *et al* 1975)

$$\mathscr{H} = -\frac{1}{2}\int_p G_0^{-1}(\boldsymbol{p})\boldsymbol{\sigma}_{\boldsymbol{p}}\cdot\boldsymbol{\sigma}_{-\boldsymbol{p}} - u\int_{q_1}\int_{q_2}\int_{q_3}(\boldsymbol{\sigma}_{q_1}\cdot\boldsymbol{\sigma}_{q_2})(\boldsymbol{\sigma}_{q_3}\cdot\boldsymbol{\sigma}_{-q_1-q_2-q_3}, \tag{10a}$$

where $\boldsymbol{\sigma}_{\boldsymbol{q}}$ is a Fourier component of the n-dimensional spin field,

$$\int_{\boldsymbol{q}} \equiv (2\pi)^{-d}\int \mathrm{d}^d q$$

and

$$G_0^{-1}(\boldsymbol{p}) = r_0 + q^2 + k^4 - gk^2; \qquad q^2 = \sum_{i=2}^{d} q_i^2; \qquad \boldsymbol{p} = k\hat{\boldsymbol{\imath}}_1 + \boldsymbol{q}. \tag{10b}$$

Note that the fourth-order term is taken to be isotropic. The critical dimensionality of this system at the Lifshitz point is $d_0 = 4{\cdot}5$, and we shall calculate η_{l4} and β_k to $O(\epsilon_l^2)$. where $\epsilon_l = d_0 - d$. As usual (Wilson 1972, Wilson and Kogut 1974), u will be set at the special value u_{oc} ($O(\epsilon_l)$) required to expose the leading scaling behaviour (cf § 4).

Now, for $t = 0$, we expect the propagator to take the form

$$G^{-1}(t=0, \delta, k, \boldsymbol{q}=0) \simeq k^{4-\eta_{l4}} Z(\hat{\mu}_t/k^{1/\nu_{l4}}, \hat{\mu}_\delta/k^{\lambda_\delta}), \tag{11}$$

where $\hat{\mu}_i \equiv \mu_i(t=0)$, and $Z(x, y)$ is analytic in x and y. Since $\hat{\mu}_\delta$ is linear in δ, we expect, in the small δ regime, to find terms in the graphical expansion of G^{-1} that match the expansion of $\delta k^{4-\eta_{l4}-\lambda_\delta}$. Such logarithmic terms can be unambiguously distinguished from those resulting from the expansion of $k^{4-\eta_{l4}-(1/\nu_{l4})}$ (which will also have $O(\delta)$ contributions).

To facilitate the graphical expansion, we write the propagator in the form

$$G^{-1}(t, \delta, k, \boldsymbol{q}) = r + q^2 + k^4 - gk^2 + \Sigma(r, g, k, \boldsymbol{q}), \tag{12a}$$

where, as usual, we introduce a renormalised mass r with the appropriate counter-terms included in the self-energy Σ. It is convenient to choose r to give the inverse susceptibility at the Lifshitz critical point (where $g = g_L$ is $O(\epsilon_l^2)$). We thus require $\Sigma(r, g_L, 0, 0) = 0$ so that

$$G^{-1}(t, \delta, k, \boldsymbol{q}) = r + q^2 + k^4 - gk^2 + \Sigma(r, g, k, \boldsymbol{q}) - \Sigma(r, g_L, 0, 0). \tag{12b}$$

Since g_L is $O(\epsilon_l^2)$ and we shall only calculate to this order, we may set $g_L = 0$ in (12*b*). Further, at $t = 0$, $r = 0$ (by construction), so to $O(\epsilon_l^2)$,

$$G^{-1}(0, \delta, k, \boldsymbol{q}) = q^2 + k^4 - gk^2 + \Sigma(0, g, k, \boldsymbol{q}) - \Sigma(0, 0, 0, 0). \tag{12c}$$

Now $g = \delta + g_L = \delta + O(\epsilon_l^2)$. Substituting in (12*c*), setting $\boldsymbol{q} = 0$, and keeping only terms to $O(\epsilon_l^2)$ finally gives

$$G^{-1}(0,\delta,k,0)=k^4-(\delta+g_L)k^2+(\Sigma(0,\delta,k,0)-\Sigma(0,0,0,0))$$
$$=k^4-(\delta+g_L)k^2+\Sigma(0,\delta,k,0)-\Sigma(0,\delta,0,0)$$
$$+\text{wavevector-independent terms.} \tag{13}$$

Thus, to identify η_{l4} and λ_δ, it will be sufficient to extract from $\Sigma(0,\delta,k,0)-\Sigma(0,\delta,0,0)$ the terms proportional to $k^4\ln k$ and $\delta k^2 \ln k$.

To $O(\epsilon_l^2,\delta)$,

$$\Sigma(0,\delta,k,0)=-32(n+2)\frac{u^2}{(2\pi)^d}\int d^d p_2[(q_2^2+k_2^4)^{-1}+3\delta k_2^2(q_2^2+k_2^4)^{-2}]I_1(\boldsymbol{p}_2+k\hat{\boldsymbol{\imath}}_1), \tag{14a}$$

where

$$I_1(\boldsymbol{p}_2+k\hat{\boldsymbol{\imath}}_1)=\frac{1}{(2\pi)^d}\int d^d p_1[(q_1^2+k_1^4)^{-1}[(\boldsymbol{q}_1+\boldsymbol{q}_2)^2+(k_1+k_2+k)^4]^{-1}. \tag{14b}$$

Since u is $O(\epsilon_l)$, the integrals in (14a) and (14b) may be evaluated precisely at the critical dimensionality $d_0=4{\cdot}5$. The calculation is eased by choosing a cylindrical Brillouin zone in which we integrate k_i from $-\Lambda_1/2$ to $+\Lambda_1/2$ and $\boldsymbol{q}_i$ over the region $0<|\boldsymbol{q}_i|<\Lambda_2$. In fact, since the integrals on $\boldsymbol{q}_i$ (in $d_0-1=3{\cdot}5$ dimensions) that we encounter are ultraviolet convergent, we set $\Lambda_2=\infty$. It then proves convenient to use the identity

$$(AB)^{-1}=\int_0^1 d\alpha[(1-\alpha)A+\alpha B]^{-2}. \tag{15}$$

Taking $A=(\boldsymbol{q}_1+\boldsymbol{q}_2)^2+(k_1+k_2+k)^4$, $B=q_1^2+k_1^4$, changing variables from $\boldsymbol{q}_1$ to $\boldsymbol{q}_1'=\boldsymbol{q}_1+\alpha\boldsymbol{q}_2$, and integrating over $0<|\boldsymbol{q}_1'|<\infty$, we obtain (Gradshteyn and Ryzhik 1965, No. 3.241.5)

$$I_1(\boldsymbol{p}_2+k\hat{\boldsymbol{\imath}}_1)=(3\sqrt{2}/16)K_{d-1}\int_{-\Lambda_1/2}^{\Lambda_1/2}dk_1\int_0^1 d\alpha[(1-\alpha)k_1^4+\alpha(k_1+k_2+k)^4$$
$$+\alpha(1-\alpha)q_2^2]^{-1/4}, \tag{16}$$

where $K_d^{-1}=2^{d-1}\pi^{d/2}\Gamma(d/2)$. A detailed analysis of (14) shows that only terms in I_1 of the form $A\ln[f(q_2,k_2+k)]$, where A is a constant, will result in contributions proportional to $k^4\ln k$ and $\delta k^2\ln k$ in $\Sigma(0,\delta,k,0)$. These can be found as follows. For $k=0$, we define

$$R\equiv[\alpha(k_1+k_2)^4+(1-\alpha)k_1^4+\alpha(1-\alpha)q_2^2]-[k_1^4+\alpha k_2^4+\alpha(1-\alpha)q_2^2]. \tag{17a}$$

We now divide the k_1 integral in (16) into segments in which

$$k_1^4+\alpha k_2^4+\alpha(1-\alpha)q_2^2\gtrless|R|, \tag{17b}$$

and define $\bar{k}_1\equiv\bar{k}_1(q_2,k_2,\alpha)$ as the largest positive value of k_1 satisfying the equation

$$k_1^4+\alpha k_2^4+\alpha(1-\alpha)q_2^2=|R|. \tag{17c}$$

(If there is no such root, we set $\bar{k}_1 = 0$.) We have

$$I_1(p_2) = (3\sqrt{2}/16)K_{d-1}\int_0^1 d\alpha\bigg(\int_{-\Lambda_1/2}^{-\bar{k}_1} dk_1[k_1^4+\alpha k_2^4+\alpha(1-\alpha)q_2^2]^{-1/4}$$
$$+\int_{-\bar{k}_1}^{\bar{k}_1} dk_1(R(k_1,k_2,\alpha))^{-1/4}+\int_{\bar{k}_1}^{\Lambda_1/2} dk_1[k_1^4+\alpha k_2^4+\alpha(1-\alpha)q_2^2]^{-1/4}$$
$$+\text{higher-order terms} \tag{18a}$$

The terms we are seeking can come only from the first and third integrals in (18a). Dividing these integrals into regions in which $k_1^4 \gtrless \alpha k_2^4+\alpha(1-\alpha)q_2^2$, we obtain the relevant ln term as

$$I_1(p_2) \approx -(3\sqrt{2}/32)K_{d-1}\ln(k_2^4+q_2^2). \tag{18b}$$

Thus, we can obtain the $k^4 \ln k$ and $\delta k^2 \ln k$ contributions of interest by considering

$$\Sigma(0,\delta,k,0)-\Sigma(0,\delta,0,0)$$
$$\approx 3\sqrt{2}(n+2)\frac{u^2}{(2\pi)^d}K_{d-1}\int d^d p_2[(k_2^4+q_2^2)^{-1}$$
$$+3\delta k_2^2(k_2^4+q_2^2)^{-2}]\{\ln[(k_2+k)^4+q_2^2]-\ln(k_2^4+q_2^2)\}. \tag{19}$$

Introducing new variables x and y by the transformation $k_2 = kx$, $q_2 = k^2 y$, we obtain from (19)

$$\Sigma(0,\delta,k,0)-\Sigma(0,\delta,0,0)$$
$$\approx (3\sqrt{2}/2\pi)(n+2)u^2K_{d-1}^2\int_{-\Lambda_1/2k}^{\Lambda_1/2k} dx\int_0^\infty d^{d-1}y[k^4(x^4+y^2)^{-1}$$
$$+3\delta k^2x^2(x^4+y^2)^{-2}]\{\ln[(x+1)^4+y^2]-\ln(x^4+y^2)\}. \tag{20}$$

Clearly ln k contributions can come only from the region in which x is large compared with unity. In this range, defining $y \equiv x^2 v$, $x^4+y^2 = x^4(1+v^2) \equiv x^4 z$, $w \equiv (x+1)^4 - x^4$, we make the expansion

$$\ln[(x+1)^4+y^2]-\ln(x^4+y^2)$$
$$= \ln(1+w/x^4z) = w/x^4z - w^2/2x^8z^2 + w^3/3x^{12}z^3 - w^4/4x^{16}z^4 + \ldots. \tag{21}$$

Collecting terms proportional to x^{-1}, we obtain the contributions we are seeking from

$$\Sigma(0,\delta,k,0)-\Sigma(0,\delta,0,0)$$
$$\approx (3\sqrt{2}/\pi)(n+2)u^2K_{d-1}^2\int_1^{\Lambda_1/2k}(dx/x)$$
$$\times\int_0^\infty dv\, v^{2\cdot 5}[(z^{-2}-34z^{-3}+96z^{-4}-64z^{-5})k^4+3\delta(6z^{-3}-8z^{-4})k^2]. \tag{22}$$

The remaining integrals are straightforward and yield

$$\Sigma(0,\delta,k,0)-\Sigma(0,\delta,0,0) \approx \tfrac{9}{32}(n+2)u^2K_{d-1}^2(k^4\ln k - 8\delta k^2\ln k). \tag{23}$$

4. Evaluation of the coupling constant

In order to obtain the exponents η_{l4} and β_k from (23), we must choose the correct coupling constant $u = u_{oc}$ to $O(\epsilon_l)$. This is done (Wilson 1972, Wilson and Kogut 1974) by requiring that the n-point vertex functions have the scaling property

$$\Gamma^{(n)}(p_1, \ldots, p_n; t) \approx \zeta^n b^{-(d-1)} a^{-1} \Gamma^{(n)}(\bar{p}_1, \ldots, \bar{p}_n; a^{1/\nu_{l4}} t), \tag{24a}$$

where $\bar{p}_i$ are the rescaled momentum components. Setting the momenta to zero, $a = t^{-\nu_{l4}}$, and, to $O(\epsilon_l)$, $b = a^2$, $\zeta^2 = a^{2d+3}$ (Hornreich *et al* 1975), (24a) becomes

$$\Gamma^{(n)}(0; t) \approx t^{-\nu_{l4}[(n-2)d+(3/2)n+1]} \tag{24b}$$

or, again to $O(\epsilon_l)$

$$\Gamma^{(n)}(0; r) \approx r^{-[(n-2)d+(3/2)n+1]/4}. \tag{24c}$$

Thus the amputated renormalised four-point vertex function should satisfy

$$u_R \equiv \Gamma^{(4)}(0; r)/(\Gamma^{(2)}(0; r))^4 \approx r^{-(2d-9)/4} \approx r^{\epsilon_l/2}. \tag{24d}$$

To determine u_{oc}, we match the expansion of (24d),

$$u_R = \text{constant } (1 + \tfrac{1}{2}\epsilon_l \ln r), \tag{25a}$$

to the result of the graphical expansion,

$$u_R = u_{oc}\left(1 - 4u_{oc}(n+2) \int d^d p (r + q^2 + k^4)^{-2}\right). \tag{25b}$$

Using the same cylindrical Brillouin zone employed in § 3, we obtain

$$u_{oc} = 2\sqrt{2}\epsilon_l/3(n+8)K_{d-1}. \tag{26}$$

5. The critical exponents

Substituting (23) and (26) into (13) and exponentiating gives

$$G^{-1}(0, \delta, k, 0) \approx k^{4-\eta_{l4}} - \delta k^{2-\eta_\phi}, \tag{27a}$$

with

$$\eta_{l4} = -[(n+2)/4(n+8)^2]\epsilon_l^2, \tag{27b}$$

$$\eta_\phi = -[2(n+2)/(n+8)^2]\epsilon_l^2. \tag{27c}$$

Matching (27a) to (11), we have

$$\beta_k = 1/\lambda_\delta = \phi/\nu_{l4} = (2 + \eta_\phi - \eta_{l4})^{-1} = \frac{1}{2} + \frac{7}{16}\frac{(n+2)}{(n+8)^2}\epsilon_l^2 + O(\epsilon_l^3). \tag{27d}$$

For the isotropic Lifshitz point, a straightforward calculation yields

$$\eta_\phi = 15\eta_{l4} = -\frac{9}{4}\frac{(n+2)}{(n+8)^2}\epsilon_\alpha^2 + O(\epsilon_\alpha^3); \qquad \epsilon_\alpha \equiv 8 - d. \tag{28}$$

The results cited in (27) and (28) are in agreement with those found independently by Mukamel (1977) for general m in a renormalisation-group calculation. In $d = 3$,

$\epsilon_l = 1{\cdot}5$ and substituting into (27) gives for the physically relevant uniaxial case $\eta_{l4} = -0{\cdot}02$ and $\beta_k = 0{\cdot}54$ for $n = 1$ and 2. The corrections to the mean-field values ($\eta_{l4} = 0$, $\beta_k = \frac{1}{2}$) are quite small, indicating that it is unlikely that either η_{l4} or β_k will differ significantly from its mean-field value in real systems.

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